A RECTANGULAR LATTICE DESIGN FOR POTATO BREEDING EXPERIMENTS *

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Since potatoes are sprayed in units of 12 rows and since the outside two rows serve as guard rows, only units of 10 rows of uniform material are available for experimental work. Therefore, it is necessary to use complete or incomplete blocks of size ten rows or else to use more than one series of ten rows if uniformity within complete or incomplete blocks is to be maintained. It was decided to use incomplete blocks of 10 single-row plots in a potato breeding experiment involving 10 check varieties and 190 entries, which were test-cross progenies, in six replicates. The parameters of the incomplete block design used are v=200, k=10, r=6, and b=6(20)=120.

The purposes of this paper are:

i) to present the statistical analysis for triple rectangular lattices of $v=2k^2$ treatments in b=2rk incomplete blocks of k treatments with 3q=r replicates,

ii) to present results from two potato experiments grown in New York in 1958, and

iii) to present a statistical analysis for partitioning the treatment degrees of freedom in estimating general and specific combining ability.

The Design

The experimental design used consisted of the following 3 basic arrangements (the treatments are designated as ijk with i=1 or 2 and j, k=1,2,•••,k):

Block No.	Arrangement I				
1	111	112	113	•••	11k
2	121	122	123		12k
3	131	132	133		13k
⋮	⋮				⋮
k	1k1	1k2	1k3		1kk
k+1	211	212	213		21k
⋮	⋮				⋮
2k	2k1	2k2	2k3	•••	2kk

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Block No.	Arrangement II				
1	111	121	131	...	1k1
2	112	122	132		1k2
3	113	123	133		1k3
⋮	⋮				⋮
k	11k	12k	13k		1kk
k+1	211	221	231		2k1
⋮	⋮				⋮
2k	21k	22k	23k	...	2kk

Block No.	Arrangement III				
1	111	122	133	...	1kk
2	112	123	134		1k1
3	113	124	135		1k2
⋮	⋮				⋮
k	11k	121	132		1kk-1
k+1	211	222	233		2kk
⋮	⋮				⋮
2k	21k	221	232	...	2kk-1

The groups of treatments in the incomplete blocks above were randomly allotted to the incomplete blocks of size 10 single row plots in the complete block of 200 single row plots. Then, each treatment was randomly allotted to the plots within each incomplete block. That is, the usual randomization procedure for one-restrictional incomplete block design was used [1,3]. The same three arrangements were used for replicates 4, 5, and 6 but with a different randomization.

The experimental design above completely confounds one treatment degree of freedom (treatments 111 to 1kk versus treatments 211 to 2kk) with differences among incomplete blocks. This could have been avoided by using the following three arrangements (for k even):

Block No.	Arrangement X = Arrangement I				
1	111	112	113	...	11k
2	121	122	123		12k
3	131	132	133		13k
⋮	⋮				⋮
k	1k1	1k2	1k3		1kk
k+1	211	212	213		21k
⋮	⋮				⋮
2k	2k1	2k2	2k3	...	2kk

Block No.	Arrangement Y (obtained from II, a=k/2)							
1	111	121	...	1a1	211	221	...	2a1
2	112	122		1a2	212	222		2a2
3	113	123		1a3	213	223		2a3
⋮	⋮							⋮
k	11k	12k		1ak	21k	22k		2ak
k+1	1,a+1,1	1,a+2,1		1k1	2,a+1,1	2,a+2,1		2k1
k+2	1,a+1,2	1,a+2,2		1k2	2,a+1,2	2,a+2,2		2k2
⋮	⋮							⋮
2k	1,a+1,k	1,a+2,k	...	1kk	2,a+1,k	2,a+2,k	...	2kk

Block No.	Arrangement Z (obtained from III, a=k/2)							
1	111	122	...	1aa	2,a+1,a+1	2,a+2,a+2	...	2kk
2	112	123		1a,a+1	2,a+1,a+2	2,a+2,a+3		2k1
3	113	124		1a,a+2	2,a+1,a+3	2,a+2,a+4		2k2
	⋮							⋮
k-1	11k-1	12k		1a,a-2	2,a+1,a-1	2,a+1,a-1		2k,k-2
k	11k	121		1a,a-1	2,a+1,a	2,a+2,a+1		2k,k-1
k+1	1,a+1,1	1,a+2,2		1ka	21,a+1	22,a+2		2ak
k+2	1,a+1,2	1,a+2,3		1k,a+1	21,a+2	22,a+3		2a1
⋮	⋮							⋮
2k	1,a+1,k	1,a+2,1	...	1k,a-1	21a	22,a+1	...	2a,k-1

Since the latter arrangement was not used, the analysis is presented only for arrangements I, II, and III. The analysis for arrangements X, Y, and Z will form the subject of a later paper.

The Analysis

The normal equations for the above design are:

$$\mu: \quad \mu\, rv + v\sum_{j=1}^{r}\rho_j + r\sum_{i=1}^{v}\tau_i + k\sum_{j=1}^{r}\sum_{g=1}^{2k}\beta_{jg} = Y_{\cdot\cdot\cdot} = \text{grand total} \;;$$

τ_i: $r(\mu+\tau_i)+\Sigma\rho_j+$ sum of block effects for blocks in which ith treatment occurred $= Y_{i\cdot\cdot} =$ ith treatment total ;

$$\rho_j: \quad v(\mu+\rho_j)+\sum_i\tau_i+\sum_g\beta_{jg} = Y_{\cdot j\cdot} = \text{jth replicate total} \;;$$

β_{jg}: $k(\mu+\rho_j+\beta_{jg})+$ sum of treatment effects for treatments occurring in block jg $= Y_{\cdot jg} =$ jgth incomplete block total .

The above plus the following equations result in unique solutions for the μ, τ_i, ρ_j, and β_{jg} effects:

$$\sum_{i=1}^{2k^2}\tau_i = \sum_{j=1}^{r}\rho_j = \sum_g\beta_{jg} = 0 \;;$$

$$\sum_{i=1}^{k^2}\tau_i = \sum_{g=1}^{k}\beta_{jg} = \frac{1}{r}\sum_{j=1}^{r}\sum_{g=1}^{k}\beta_{jg} \;;$$

$$\sum_{i=k^2+1}^{2k^2}\tau_i = \sum_{g=k+1}^{2k}\beta_{jg} = \frac{1}{r}\sum_{j=1}^{r}\sum_{g=k+1}^{2k}\beta_{jg} \;.$$

The solutions for the β_{jg} are

$$\hat{\beta}_{jg} = \frac{1}{2qk^2}\left\{3kQ_{\cdot jg} - \sum_{g=1}^{k}Q_{\cdot jg}\right\} + \frac{1}{k(k+1)}\sum_{i=1}^{k^2}(\bar{y}_{i\cdot\cdot}-\bar{y})$$

for $g=1,2,\cdots,k$ and

$$\hat{\beta}_{jg} = \frac{1}{2qk^2}\left\{3kQ_{\circ jg} - \sum_{g=k+1}^{2k} Q_{\cdot jg}\right\} + \frac{1}{k(k+1)} \sum_{i=k^2+1}^{2k^2} (\bar{y}_{i\cdot\cdot} - \bar{y})$$

for $g=k+1, k+2, \cdots, 2k$. For $q=1$, the value for $Q_{\cdot jg}$ equals the incomplete block total minus the means of the varieties which appeared in block g of the jth replicate. If $q > 1$, $Q_{\circ jg}$ represents the sum of the $Q_{\cdot jg}$ values for each set of 3 basic arrangements, or it equals the sum of q incomplete block totals for a given group of treatments minus q times the mean of these varieties.

The adjusted variety means are $\hat{\mu}+\hat{\tau}_i=\bar{y}+\hat{\tau}_i$ and the $\hat{\tau}_i$ are obtained from the normal equations for the treatments or varieties. This is the variety mean adjusted for intrablock information only.

The analysis of variance for the above design follows:

Source of variation	d.f.	Sum of squares
Replicates	$r-1$	$\sum_{j=1}^{r} \frac{Y^2_{\cdot j\cdot}}{2k^2} - \frac{Y^2_{\cdot\cdot\cdot}}{2rk^2}$
Treatments (elim. incomplete blocks)	$2k^2-2$	$\sum_{i=1}^{v} \hat{\tau}_i Q_{i\cdot\cdot}$
Incomplete blocks (ign. treat.)	$6qk-r$	$\sum_{j=1}^{r}\sum_{g=1}^{2k} \frac{Y^2_{\cdot jg}}{k} - \sum_{j=1}^{r} \frac{Y^2_{\cdot j\cdot}}{2k^2}$
A=Tr. 1 to k^2 vs Tr. k^2+1 to $2k^2$	1	$[\sum_{i=1}^{k^2} Y_{i\cdot\cdot} - \sum_{i=k^2+1}^{2k^2} Y_{i\cdot\cdot}]^2/2rk^2$
A x replicates	$r-1$	usual method
Residual	$6qk-2r$	subtraction
Intrablock error	$2(k-1)(r-1)(k-1)$	subtraction
Incomplete blocks (elim. treat.)	$6qk-r-1$	see below
A x replicates	$r-1$	see below
Component (a)	$6(q-1)(k-1)$	see below
Component (b)	$6(k-1)$	see below

$Q_{i..}$ in the above analysis of variance table is equal to the total yield of the ith treatment, $Y_{i..}$, minus the sum of the means of the r incomplete blocks in which the ith treatment occurred.

The component (b) sum of squares is computed as

$$\sum_{j=1}^{3} \sum_{g=1}^{k} \beta_{jg} Q^{\cdot}_{\cdot jg}/q + \sum_{j=1}^{3} \sum_{j=k+1}^{2k} \beta_{jg} Q^{\cdot}_{\cdot jg}/q$$

where the $Q^{\cdot}_{\cdot jg}$ values represents the sum of the $Q_{\cdot jg}$ values from the q sets of 3 arrangements.

The component (a) sum of squares is the interaction of block totals $Y_{\cdot jg}$, with the q replicates which have the same basic arrangement, within sets of treatments 1 to k^2 and k^2+1 to $2k^2$.

Both the A x replicates and the component (a) mean squares have expectation $\sigma_\epsilon^2+k\sigma_\beta^2$; the expectation of the component (b) mean square is $\sigma_\epsilon^2+\frac{2}{3}k\sigma_\beta^2$. Therefore, the expected value of the blocks (eliminating treatment) mean square $=E_b$, is:

$$\frac{[r-1+6(q-1)(k-1)][\sigma_\epsilon^2+k\sigma_\beta^2]+[6(k-1)][\sigma_\epsilon^2+\frac{2}{3}k\sigma_\beta^2]}{6qk-3q-1}=\sigma_\epsilon^2+\frac{(3q-1)(2k-1)}{6qk-3q-1}k\sigma_\beta^2 .$$

The weights are estimated as:

$$w=\frac{1}{\text{Intrablock error M.S.}=E_e}$$

and

$$w'=\frac{(3q-1)(2k-1)}{(6qk-3q-1)E_b-2(k-1)E_e} .$$

The normal equations for obtaining adjusted treatment means utilizing interblock information are [2,3]:

$$\mu:\quad 6qk^2\mu(w+w')+v\sum_{j=1}^{r}\rho_j(w+w')+3q(w+w')\sum_{i=1}^{v}\tau_i+wk\sum_{j=1}^{r}\sum_{g=1}^{2k}\beta_{jg}=Y_{\cdot\cdot\cdot}(w+w')\quad ;$$

$$\rho_j:\quad v(\rho_j+\mu)(w+w')+\sum_i\tau_i(w+w')+w\sum_g\beta_{jg}=Y_{\cdot j\cdot}(w+w')\quad ;$$

β_{jg}: $kw(\mu+\rho_j+\beta_{jg})+w$ (sum of treatment effects for treatments occurring in block $jg = wY_{\circ jg}$;

$$\tau_h:\ r\mu(w+w')+(w+w')\Sigma\rho_j+rw\tau_h+\frac{w'}{k}\sum_{i=1}^{v}\tau_i\sum_{jg}\sum n_{hjg}n_{ijg}+w\sum_{jg}\sum n_{hjg}\beta_{jg}$$

$$=wY_{h\circ\bullet}+\frac{w'}{k}\sum_{jg}\sum n_{hjg}Y_{\bullet jg} \quad .$$

The above equations plus the following result in unique solutions for the effects:

$$\sum_{i=1}^{v}\tau_i=\sum_{j=1}^{r}\rho_j=\sum_{g}\beta_{jg}=0 \quad ;$$

$$\sum_{i=1}^{k^2}\tau_i=\sum_{g=1}^{k}\beta_{jg}=\frac{1}{3q}\sum_{j=1}^{r}\sum_{g=1}^{k}\beta_{jg} \quad ;$$

$$\sum_{i=k^2+1}^{2k^2}\tau_i=\sum_{g=k+1}^{2k}\beta_{jg}=\frac{1}{3q}\sum_{j=1}^{r}\sum_{g=k+1}^{2k}\beta_{jg} \quad .$$

The solutions for the τ_i are obtained from the following v equations, utilizing the above equations, the hth equation being:

$$\hat{\tau}_h\left\{w(n_{h\bullet\bullet}-\sum_{jg}\sum n^2_{hjg}/n_{\circ jg})-w'\sum_{jg}\sum\frac{n_{hjg}}{n_{\bullet jg}}\left(\frac{n_{h\bullet\bullet}}{b}-n_{hjg}\right)\right\}$$

$$-\sum_{\substack{i=1\\ \neq k}}^{v}\hat{\tau}_i\left\{w\sum_{jg}\sum\frac{n_{ijg}n_{hjg}}{n_{\circ jg}}-w'\sum_{jg}\sum\frac{n_{ijg}}{n_{\bullet jg}}\left(n_{hjg}-\frac{n_{h\bullet\bullet}}{b}\right)\right\}$$

$$=w\left\{Y_{h\bullet\bullet}-\sum_{jg}\sum n_{hjg}\bar{y}_{\bullet jg}\right\}+w'\left\{\sum_{jg}\sum n_{hjg}\bar{y}_{\bullet jg}\right.$$

$$\left.-\frac{n_{h\bullet\bullet}}{b}\sum_{jg}\sum\bar{y}_{\bullet jg}\right\}=Z_{h\circ\circ} \quad .$$

Solving these equations we obtain the solutions in matrix form as:

$$\underline{\hat{\tau}} = N^{-1}\underline{Z}$$

where N^{-1} is the inverse matrix of coefficients for the variance-covariance matrix recovering interblock information.

An Example

To illustrate the estimation of effects and the computations in the analysis of variance the following example was constructed for ease of computation from the parameters v=18, k=3, r=6, b=36 and

$\mu = 10$	$\beta_{11} = -1$	$\beta_{21} = 0$	$\beta_{31} = -1$	$\beta_{41} = -1$
$\rho_1 = -1$	$\beta_{12} = -2$	$\beta_{22} = 0$	$\beta_{32} = 1$	$\beta_{42} = -2$
$\rho_2 = -2$	$\beta_{13} = 2$	$\beta_{23} = -1$	$\beta_{33} = -1$	$\beta_{43} = 2$
$\rho_3 = 3$	$\beta_{14} = -2$	$\beta_{24} = -1$	$\beta_{34} = 1$	$\beta_{44} = -2$
$\rho_4 = 3$	$\beta_{15} = 2$	$\beta_{25} = 3$	$\beta_{35} = 1$	$\beta_{45} = 2$
$\rho_5 = -2$	$\beta_{16} = 1$	$\beta_{26} = -1$	$\beta_{36} = -1$	$\beta_{46} = 1$
$\rho_6 = -1$				
$\beta_{51} = 0$	$\beta_{61} = -1$	$\tau_1 = -1$	$\tau_7 = 4$	$\tau_{13} = 0$
$\beta_{52} = 0$	$\beta_{62} = 1$	$\tau_2 = -2$	$\tau_8 = 3$	$\tau_{14} = 0$
$\beta_{53} = -1$	$\beta_{63} = -1$	$\tau_3 = -3$	$\tau_9 = 2$	$\tau_{15} = 0$
$\beta_{54} = -1$	$\beta_{64} = 1$	$\tau_4 = -4$	$\tau_{10} = 1$	$\tau_{16} = 0$
$\beta_{55} = 3$	$\beta_{65} = -1$	$\tau_5 = -5$	$\tau_{11} = 0$	$\tau_{17} = 0$
$\beta_{56} = -1$	$\beta_{66} = 1$	$\tau_6 = 5$	$\tau_{12} = 0$	$\tau_{18} = 0$

$\epsilon_{111} = -1$	$\epsilon_{121} = 1$	$\epsilon_{1014} = -1$	$\epsilon_{1024} = 1$
$\epsilon_{211} = 1$	$\epsilon_{222} = -1$	$\epsilon_{1114} = 1$	$\epsilon_{1125} = -1$
$\epsilon_{412} = 1$	$\epsilon_{421} = -1$	$\epsilon_{1315} = 1$	$\epsilon_{1324} = -1$
$\epsilon_{512} = -1$	$\epsilon_{522} = 1$	$\epsilon_{1415} = -1$	$\epsilon_{1425} = 1$

Yields for schematic arrangement; treatment numbers in parentheses

Replicate number

Block number	1 (arrangement I)			Block sum	Block number	2 (arrangement II)			Block sum
11	(1)	(2)	(3)		21	(1)	(4)	(7)	
	6	7	5	18		8	3	12	23
12	(4)	(5)	(6)		22	(2)	(5)	(8)	
	4	1	12	17		5	4	11	20
13	(7)	(8)	(9)		23	(3)	(6)	(9)	
	15	14	13	42		4	12	9	25
14	(10)	(11)	(12)		24	(10)	(13)	(16)	
	7	8	7	22		9	6	7	22
15	(13)	(14)	(15)		25	(11)	(14)	(17)	
	12	10	11	33		10	12	11	33
16	(16)	(17)	(18)		26	(12)	(15)	(18)	
	10	10	10	30		7	7	7	21
	Replicate 1 total =			162		Replicate 2 total =			144

Block number	3 (arrangement III)			Block sum	Block number	4 (arrangement II)			Block sum
31	(1)	(5)	(9)		41	(1)	(4)	(7)	
	11	7	14	32		11	8	16	35
32	(2)	(6)	(7)		42	(2)	(5)	(8)	
	12	19	18	49		9	6	14	29
33	(3)	(4)	(8)		43	(3)	(6)	(9)	
	9	8	15	32		12	20	17	49
34	(10)	(14)	(18)		44	(10)	(13)	(16)	
	15	14	14	43		12	11	11	34
35	(11)	(15)	(16)		45	(11)	(14)	(17)	
	14	14	14	42		15	15	15	45
36	(12)	(13)	(17)		46	(12)	(15)	(18)	
	12	12	12	36		14	14	14	42
	Replicate 3 total =			234		Replicate 4 total =			234

Block number	5 (arrangement III)			Block sum	Block number	6 (arrangement I)			Block sum
51	(1)	(5)	(9)		61	(1)	(2)	(3)	
	7	3	10	20		7	6	5	18
52	(2)	(6)	(7)		62	(4)	(5)	(6)	
	6	13	12	31		6	5	15	26
53	(3)	(4)	(8)		63	(7)	(8)	(9)	
	4	3	10	17		12	11	10	33
54	(10)	(14)	(18)		64	(10)	(11)	(12)	
	8	7	7	22		11	10	10	31
55	(11)	(15)	(16)		65	(13)	(14)	(15)	
	11	11	11	33		8	8	8	24
56	(12)	(13)	(17)		66	(16)	(17)	(18)	
	7	7	7	21		10	10	10	30
	Replicate 5 total =			144		Replicate 6 total =			162

First let's consider the analysis of replicates 1, 2, and 3. The normal equations for this example are:

μ: $\quad 3(18)\hat{\mu}+0+0+0=Y_{\circ\circ\circ}=540=162+144+234$

ρ_1: $\quad 18(\hat{\mu}+\hat{\rho}_1)+0+0=Y_{\cdot 1\cdot}=162$

ρ_2: $\quad 18(\hat{\mu}+\hat{\rho}_2)+0+0=Y_{\circ 2\circ}=144$

ρ_3: $\quad 18(\hat{\mu}+\hat{\rho}_3)+0+0=Y_{\circ 3\circ}=234$

τ_1: $\quad 3(\hat{\mu}+\hat{\tau}_1)+0+\hat{\beta}_{11}+\hat{\beta}_{21}+\hat{\beta}_{31}=Y_{1\circ\cdot}=6+8+11=25$

τ_2: $\quad 3(\hat{\mu}+\hat{\tau}_2)+0+\hat{\beta}_{11}+\hat{\beta}_{22}+\hat{\beta}_{32}=Y_{2\cdot\cdot}=24$

τ_3: $\quad 3(\hat{\mu}+\hat{\tau}_3)+0+\hat{\beta}_{11}+\hat{\beta}_{23}+\hat{\beta}_{33}=Y_{3\circ\cdot}=18$

τ_4: $\quad 3(\hat{\mu}+\hat{\tau}_4)+0+\hat{\beta}_{12}+\hat{\beta}_{21}+\hat{\beta}_{33}=Y_{4\circ\cdot}=15$

τ_5: $\quad 3(\hat{\mu}+\hat{\tau}_5)+0+\hat{\beta}_{12}+\hat{\beta}_{22}+\hat{\beta}_{31}=Y_{5\cdot\cdot}=12$

τ_6: $\quad 3(\hat{\mu}+\hat{\tau}_6)+0+\hat{\beta}_{12}+\hat{\beta}_{23}+\hat{\beta}_{32}=Y_{6\circ\cdot}=43$

τ_7: $\quad 3(\hat{\mu}+\hat{\tau}_7)+0+\hat{\beta}_{13}+\hat{\beta}_{21}+\hat{\beta}_{32}=Y_{7\cdot\cdot}=45$

τ_8 : $3(\hat{\mu}+\hat{\tau}_8)+0+\hat{\beta}_{13}+\hat{\beta}_{22}+\hat{\beta}_{33}=Y_{8..}=40$

τ_9 : $3(\hat{\mu}+\hat{\tau}_9)+0+\hat{\beta}_{13}+\hat{\beta}_{23}+\hat{\beta}_{31}=Y_{9..}=36$

τ_{10}: $3(\hat{\mu}+\hat{\tau}_{10})+0+\hat{\beta}_{14}+\hat{\beta}_{24}+\hat{\beta}_{34}=Y_{10..}=31$

τ_{11}: $3(\hat{\mu}+\hat{\tau}_{11})+0+\hat{\beta}_{14}+\hat{\beta}_{25}+\hat{\beta}_{35}=Y_{11..}=32$

τ_{12}: $3(\hat{\mu}+\hat{\tau}_{12})+0+\hat{\beta}_{14}+\hat{\beta}_{26}+\hat{\beta}_{36}=Y_{12..}=26$

τ_{13}: $3(\hat{\mu}+\hat{\tau}_{13})+0+\hat{\beta}_{15}+\hat{\beta}_{24}+\hat{\beta}_{36}=Y_{13..}=30$

τ_{14}: $3(\hat{\mu}+\hat{\tau}_{14})+0+\hat{\beta}_{15}+\hat{\beta}_{25}+\hat{\beta}_{34}=Y_{14..}=36$

τ_{15}: $3(\hat{\mu}+\hat{\tau}_{15})+0+\hat{\beta}_{15}+\hat{\beta}_{26}+\hat{\beta}_{35}=Y_{15..}=32$

τ_{16}: $3(\hat{\mu}+\hat{\tau}_{16})+0+\hat{\beta}_{16}+\hat{\beta}_{24}+\hat{\beta}_{35}=Y_{16..}=31$

τ_{17}: $3(\hat{\mu}+\hat{\tau}_{17})+0+\hat{\beta}_{16}+\hat{\beta}_{25}+\hat{\beta}_{36}=Y_{17..}=33$

τ_{18}: $3(\hat{\mu}+\hat{\tau}_{18})+0+\hat{\beta}_{16}+\hat{\beta}_{26}+\hat{\beta}_{34}=Y_{18..}=31$

β_{11}: $3(\hat{\mu}+\hat{\rho}_1+\hat{\beta}_{11})+\hat{\tau}_1+\hat{\tau}_2+\hat{\tau}_3=Y_{.11}=18$

β_{12}: $3(\hat{\mu}+\hat{\rho}_1+\hat{\beta}_{12})+\hat{\tau}_4+\hat{\tau}_5+\hat{\tau}_6=Y_{.12}=17$

β_{13}: $3(\hat{\mu}+\hat{\rho}_1+\hat{\beta}_{13})+\hat{\tau}_7+\hat{\tau}_8+\hat{\tau}_9=Y_{.13}=42$

β_{14}: $3(\hat{\mu}+\hat{\rho}_1+\hat{\beta}_{14})+\hat{\tau}_{10}+\hat{\tau}_{11}+\hat{\tau}_{12}=Y_{.14}=22$

β_{15}: $3(\hat{\mu}+\hat{\rho}_1+\hat{\beta}_{15})+\hat{\tau}_{13}+\hat{\tau}_{14}+\hat{\tau}_{15}=Y_{.15}=33$

β_{16}: $3(\hat{\mu}+\hat{\rho}_1+\hat{\beta}_{16})+\hat{\tau}_{16}+\hat{\tau}_{17}+\hat{\tau}_{18}=Y_{.16}=30$

β_{21}: $3(\hat{\mu}+\hat{\rho}_2+\hat{\beta}_{21})+\hat{\tau}_1+\hat{\tau}_4+\hat{\tau}_7=Y_{.21}=23$

β_{22}: $3(\hat{\mu}+\hat{\rho}_2+\hat{\beta}_{22})+\hat{\tau}_2+\hat{\tau}_5+\hat{\tau}_8=Y_{.22}=20$

β_{23}: $3(\hat{\mu}+\hat{\rho}_2+\hat{\beta}_{23})+\hat{\tau}_3+\hat{\tau}_6+\hat{\tau}_9=Y_{.23}=25$

β_{24}: $3(\hat{\mu}+\hat{\rho}_2+\hat{\beta}_{24})+\hat{\tau}_{10}+\hat{\tau}_{13}+\hat{\tau}_{16}=Y_{.24}=22$

β_{25}: $3(\hat{\mu}+\hat{\rho}_2+\hat{\beta}_{25})+\hat{\tau}_{11}+\hat{\tau}_{14}+\hat{\tau}_{17}=Y_{.25}=33$

β_{26}: $3(\hat{\mu}+\hat{\rho}_2+\hat{\beta}_{26})+\hat{\tau}_{12}+\hat{\tau}_{15}+\hat{\tau}_{18}=Y_{.26}=21$

β_{31}: $3(\hat{\mu}+\hat{\rho}_3+\hat{\beta}_{31})+\hat{\tau}_1+\hat{\tau}_5+\hat{\tau}_9=Y_{.31}=32$

β_{32}: $3(\hat{\mu}+\hat{\rho}_3+\hat{\beta}_{32})+\hat{\tau}_2+\hat{\tau}_6+\hat{\tau}_7=Y_{.32}=49$

β_{33}: $3(\hat{\mu}+\hat{\rho}_3+\hat{\beta}_{33})+\hat{\tau}_3+\hat{\tau}_4+\hat{\tau}_8=Y_{.33}=32$

β_{34}: $3(\hat{\mu}+\hat{\rho}_3+\hat{\beta}_{34})+\hat{\tau}_{10}+\hat{\tau}_{14}+\hat{\tau}_{18}=Y_{.34}=43$

β_{35}: $3(\hat{\mu}+\hat{\rho}_3+\hat{\beta}_{35})+\hat{\tau}_{11}+\hat{\tau}_{15}+\hat{\tau}_{16}=Y_{.35}=42$

β_{36}: $3(\hat{\mu}+\hat{\rho}_3+\hat{\beta}_{36})+\hat{\tau}_{12}+\hat{\tau}_{13}+\hat{\tau}_{17}=Y_{.36}=36$

$$\sum_{i=1}^{9} \hat{\tau}_i = \sum_{g=1}^{3} \hat{\beta}_{jg} = -1 \quad ;$$

$$\sum_{i=10}^{18} \hat{\tau}_i = \sum_{g=4}^{6} \hat{\beta}_{jg} = +1 \quad ;$$

$\bar{y}=10$, $\bar{y}_{.1}=9$, $\bar{y}_{.2}=8$, $\bar{y}_{.3}=13$

$Q_{.11} = 18-1/3(25+24+18)-3(9-10) = -4/3$

$Q_{.12} = 17-1/3(15+12+43)-3(9-10) = -10/3$

$Q_{.13} = 42-1/3(45+40+36)-3(9-10) = 14/3$

$Q_{.14} = 22-1/3(31+32+26)-3(9-10) = -14/3$

$Q_{.15} = 33-1/3(30+36+32)-3(9-10) = 10/3$

$Q_{.16} = 30-1/3(31+33+31)-3(9-10) = 4/3$

$Q_{.21} = 23-1/3(85)-3(8-10) = 2/3$

$Q_{.22} = 20-1/3(76)-3(8-10) = 2/3$

$Q_{.23} = 25-1/3(97)-3(8-10) = -4/3$

$Q_{.24} = 22-1/3(92)-3(8-10) = -8/3$

$Q_{.25} = 33-1/3(101)-3(8-10) = 16/3$

$Q_{.26} = 21-1/3(89)-3(8-10) = -8/3$

$Q_{.31} = 32-3(13-10)-73/3 = -4/3$

$Q_{.32} = 49-3(13-10)-112/3 = 8/3$

$Q_{.33} = 32-3(13-10)-73/3 = -4/3$

$Q_{.34} = 43-3(13-10)-98/3 = 4/3$

$Q_{.35} = 42-3(13-10)-95/3 = 4/3$

$Q_{.36} = 36-3(13-10)-89/3 = -8/3$

$Q_{1..} = 25-1/3(18+23+32) = 2/3$

$Q_{2..} = 24-1/3(18+20+49) = -5$

$Q_{3..} = 18-1/3(18+25+32) = -7$

$Q_{4..} = 15-1/3(17+23+32) = -9$

$Q_{5..} = 12-1/3(17+20+32) = -11$

$Q_{6..} = 43-1/3(17+25+49) = 38/3$

$Q_{7..} = 45-1/3(42+23+49) = 7$

$Q_{8..} = 40-1/3(42+20+32) = 26/3$

$Q_{9..} = 36-1/3(42+25+32) = 3$

$Q_{10..} = 31-1/3(22+22+43) = 2$

$Q_{11..} = 32-1/3(22+33+42) = -1/3$

$Q_{12..} = 26-1/3(22+21+36) = -1/3$

$Q_{13..} = 30-1/3(33+22+36) = -1/3$

$Q_{14..} = 36-1/3(33+33+43) = -1/3$

$Q_{15..} = 32-1/3(33+21+42) = 0$

$Q_{16..} = 31-1/3(30+22+42) = -1/3$

$Q_{17..} = 33-1/3(30+33+36) = 0$

$Q_{18..} = 31-1/3(30+21+43) = -1/3$

$\hat{\beta}_{11} = 1/18\left\{3(3)(-4/3)\right\}+1/12(258/3-90)=-12/18-12/36 = -1$

$\hat{\beta}_{12} = 1/18\left\{3(3)(-10/3)\right\}-1/3 = -2$

$\hat{\beta}_{13} = 1/2(14/3)-1/3 = 2$

$\hat{\beta}_{14} = 1/2(-14/3)+1/3 = -2$

$\hat{\beta}_{15} = 1/2(10/3)+1/3 = 2$

$\hat{\beta}_{16} = 1/2(4/3)+1/3 = 1$

$\hat{\beta}_{21} = 1/2(2/3)-1/3 = 0$

$\hat{\beta}_{22} = 1/2(2/3)-1/3 = 0$

$\hat{\beta}_{23} = 1/2(-4/3)-1/3 = -1$

$\hat{\beta}_{24} = 1/2(-8/3)+1/3 = -1$

$\hat{\beta}_{25} = 1/2(16/3)+1/3 = 3$

$\hat{\beta}_{26} = 1/2(-8/3)+1/3 = -1$

$\hat{\beta}_{31} = 1/2(-4/3)-1/3 = -1$

$\hat{\beta}_{32} = 1/2(8/3)-1/3 = 1$

$\hat{\beta}_{33} = 1/2(-4/3)-1/3 = -1$

$\hat{\beta}_{34} = 1/2(4/3)+1/3 = 1$

$\hat{\beta}_{35} = 1/2(4/3)+1/3 = 1$

$\hat{\beta}_{36} = 1/2(-8/3)+1/3 = -1$

$$\hat{\tau}_1 = \bar{y}_{1..}-\bar{y}-1/3(\hat{\beta}_{11}+\hat{\beta}_{21}+\hat{\beta}_{31})=\bar{y}_{1..}-\bar{y}-1/3\left\{1/2(Q_{.11}+Q_{.21}+Q_{.31})\right.$$

$$\left.-1/18\sum_{j=1}^{3}\sum_{g=1}^{3}Q_{.jg}\right\}-1/12\sum_{i=1}^{9}(\bar{y}_{i..}-\bar{y})$$

$$=25/3-10-1/6(-4/3+2/3-4/3)+0-1/12(-12/3) = -1$$

$\hat{\tau}_2 = 8-10-1/3(-1+0+1) = -2$

$\hat{\tau}_3 = 6-10-1/3(-1-1-1) = -3$

$\hat{\tau}_4 = 5-10-1/3(-2+0-1) = -4$

$\hat{\tau}_5 = 4-10-1/3(-2+0-1) = -5$

$\hat{\tau}_6 = 43/3-10-1/3(-2-1+1) = 5$

$\hat{\tau}_7 = 15-10-1/3(2+0+1) = 4$

$\hat{\tau}_8 = 40/3-10-1/3(2+0-1) = 3$

$\hat{\tau}_9 = 12-10-1/3(2-1-1) = 2$

$\hat{\tau}_{10} = 31/3-10-1/3(-2-1+1) = 1$

$\hat{\tau}_{11} = 32/3-10-1/3(-2+3+1) = 0$

$\hat{\tau}_{12} = 26/3-10-1/3(-2-1-1) = 0$

$\hat{\tau}_{13} = 10-10-1/3(2-1-1) = 0$

$\hat{\tau}_{14} = 12-10-1/3(2+3+1) = 0$

$\hat{\tau}_{15} = 32/3-10-1/3(2-1+1) = 0$

$\hat{\tau}_{16} = 31/3-10-1/3(1-1+1) = 0$

$\hat{\tau}_{17} = 11-10-1/3(1+3-1) = 0$

$\hat{\tau}_{18} = 31/3-10-1/3(1-1+1) = 0$

With the above estimates of $\hat{\mu}$, $\hat{\rho}_j$, $\hat{\tau}_i$ and $\hat{\beta}_{jg}$ agreeing with the parameters used to set up the numerical example, we now proceed with computing the following sums of squares in the analysis of variance:

Total with 53 d.f.

$$\sum_{i=1}^{18} \sum_{j=1}^{3} Y^2_{ijg} - \frac{Y^2_{\cdots}}{54} = 6188 - \frac{540^2}{54} = 6188 - 5400 = 788 \quad .$$

Replicate with 2 d.f.

$$\sum_{j=1}^{3} \frac{Y^2_{\cdot j \cdot}}{18} - \frac{Y^2_{\cdots}}{54} = 252 \quad .$$

Treatments (ignoring block effects) with 17 d.f.

$$\sum_{i=1}^{18} \frac{Y^2_{i\cdot\cdot}}{3} - \frac{Y^2_{\cdots}}{54} = 452 \quad .$$

Blocks (ignoring treatment effects) with 15 d.f.

$$\sum_{j=1}^{3} \left\{ \sum_{g=1}^{6} \frac{Y^2_{\cdot jg}}{k=3} - \frac{Y^2_{\cdot j \cdot}}{18} \right\} = 5925.333 - 5652 = \frac{820}{3} = 273.333 \quad .$$

Treatments 1 to 9 vs. treatments 10 to 18 with 1 d.f.

$$\frac{(258-282)^2}{3(9)(1+1)} = \frac{32}{3} \quad .$$

Treatments 1 to 9 vs. treatments 10 to 18 x replicates with 2 d.f.

$$\frac{(77-85)^2}{18} + \frac{(68-76)^2}{18} + \frac{(113-121)^2}{18} - \frac{(258-282)^2}{54} = 0 \quad .$$

Blocks (eliminating treatment effects) within groups with 12 d.f.

$$\sum_{j=1}^{3} \left\{ \sum_{g=1}^{3} \hat{\beta}_{jg} Q_{\cdot jg} - \sum_{g=1}^{3} \hat{\beta}_{jg} \sum_{g=1}^{3} Q_{\cdot jg}/3 + \sum_{g=4}^{6} \hat{\beta}_{jg} Q_{\cdot jg} - \sum_{g=4}^{6} \hat{\beta}_{jg} \sum_{g=4}^{6} Q_{\cdot jg}/3 \right\} = 68 \quad .$$

Treatment (eliminating block effects) with 16 d.f.

$$\Sigma \hat{\tau}_i Q_{i\cdot\cdot} = \frac{740}{3} = 246.667 \quad .$$

Combining these sums of squares in an analysis of variance table we obtain:

Source of variation	d.f.		s.s.		m.s.
Total (uncorrected)	54		6188		--
Correction for the mean	1		5400		--
Total (corrected for the mean)	53		788		--
Replicates	2		252		--
Treatments (ignoring blocks)	17		452		--
1 to 9 vs. 10 to 18		1		10.667	--
Remainder		16		441.333	--
Replicates x treatments	34		84		
Blocks within treatment		12		68	17/3=5.667
Groups (elim. treatment effects)					
Treatment groups x reps.		2		0	0
Residual = intrablock error		20		16	4/5=.8
Treatment (elim. block effects)	17		772/3		--
1 to 9 vs. 10 to 18		1		32/3	10.667
Treatments within groups		16		740/3	15.417

The residual or intrablock error sum of squares equals 16 as it should since 16 plus or minus ones were inserted in the table of yields. As a further check, the correction for disproportion is computed as $452-772/3=584/3$ and $273\frac{1}{3}-(68+10\frac{2}{3})=\frac{820-236}{3}=\frac{584}{3}$. Thus, everything checks for this example. It should be remembered that one treatment degree of freedom is completely entangled or confounded with blocks and that this should be removed before making corrections for disproportion. We could do this as follows: The among treatments (eliminating block effects) within groups of k^2 sum of squares with $2(k^2-1)=16$ degrees of freedom is $452-\frac{32}{3}-\frac{584}{3}=\frac{1324}{3}-\frac{584}{3}=\frac{740}{3}$; the among blocks (eliminating treatment effects) within groups of k^2 sum of squares is $\frac{820-32-0}{3}-\frac{584}{3}=\frac{204}{3}=68$.

The intrablock analysis for the six replicates will be given now; then, the analysis with recovery of interblock information and the computation of the various variances will be given. As before we write out the normal equations with an additional set of equations and obtain unique solutions for the effects. The normal equations follow:

$$\mu: \quad 6(18)\hat{\mu}+18\sum_{j=1}^{6}\hat{\rho}_j+6\sum_{i=1}^{18}\hat{\tau}_i+3\sum_{j=1}^{6}\sum_{g=1}^{6}\hat{\beta}_{jg}=6(18)\hat{\mu}+0+0+0=Y_{\cdot\cdot\cdot}=1080 \quad ;$$

$$\rho_j: \quad 18(\hat{\mu}+\hat{\rho}_j)+\sum_{i=1}^{18}\hat{\tau}_i+\sum_{g=1}^{6}\hat{\beta}_{jg}=18(\hat{\mu}+\hat{\rho}_j)=Y_{\cdot j\cdot}=\text{replicate total} \quad ;$$

$$\tau_1: \quad 6(\hat{\mu}+\hat{\tau}_1)+0+\hat{\beta}_{11}+\hat{\beta}_{21}+\hat{\beta}_{31}+\hat{\beta}_{41}+\hat{\beta}_{51}+\hat{\beta}_{61}=Y_{1\cdot\cdot}=50$$

$$\tau_2: \quad 6(\hat{\mu}+\hat{\tau}_2)+0+\hat{\beta}_{11}+\hat{\beta}_{22}+\hat{\beta}_{32}+\hat{\beta}_{42}+\hat{\beta}_{52}+\hat{\beta}_{61}=Y_{2\cdot\cdot}=45$$

$$\tau_3: \quad 6(\hat{\mu}+\hat{\tau}_3)+0+\hat{\beta}_{11}+\hat{\beta}_{23}+\hat{\beta}_{33}+\hat{\beta}_{43}+\hat{\beta}_{53}+\hat{\beta}_{61}=Y_{3\cdot\cdot}=39$$

$$\tau_4: \quad 6(\hat{\mu}+\hat{\tau}_4)+0+\hat{\beta}_{12}+\hat{\beta}_{21}+\hat{\beta}_{33}+\hat{\beta}_{41}+\hat{\beta}_{53}+\hat{\beta}_{62}=Y_{4\cdot\cdot}=32$$

$$\tau_5: \quad 6(\hat{\mu}+\hat{\tau}_5)+0+\hat{\beta}_{12}+\hat{\beta}_{22}+\hat{\beta}_{31}+\hat{\beta}_{42}+\hat{\beta}_{51}+\hat{\beta}_{62}=Y_{5\cdot\cdot}=26$$

$$\tau_6: \quad 6(\hat{\mu}+\hat{\tau}_6)+0+\hat{\beta}_{12}+\hat{\beta}_{23}+\hat{\beta}_{32}+\hat{\beta}_{43}+\hat{\beta}_{52}+\hat{\beta}_{62}=Y_{6\cdot\cdot}=91$$

$$\tau_7: \quad 6(\hat{\mu}+\hat{\tau}_7)+0+\hat{\beta}_{13}+\hat{\beta}_{21}+\hat{\beta}_{32}+\hat{\beta}_{41}+\hat{\beta}_{52}+\hat{\beta}_{63}=Y_{7\cdot\cdot}=85$$

$$\tau_8: \quad 6(\hat{\mu}+\hat{\tau}_8)+0+\hat{\beta}_{13}+\hat{\beta}_{22}+\hat{\beta}_{33}+\hat{\beta}_{42}+\hat{\beta}_{53}+\hat{\beta}_{63}=Y_{8\cdot\cdot}=75$$

$$\tau_9: \quad 6(\hat{\mu}+\hat{\tau}_9)+0+\hat{\beta}_{13}+\hat{\beta}_{23}+\hat{\beta}_{31}+\hat{\beta}_{43}+\hat{\beta}_{51}+\hat{\beta}_{63}=Y_{9\cdot\cdot}=73$$

τ_{10}: $6(\hat{\mu}+\hat{\tau}_{10})+0+\hat{\beta}_{14}+\hat{\beta}_{24}+\hat{\beta}_{34}+\hat{\beta}_{44}+\hat{\beta}_{54}+\hat{\beta}_{64}=Y_{10..} = 62$

τ_{11}: $6(\hat{\mu}+\hat{\tau}_{11})+0+\hat{\beta}_{14}+\hat{\beta}_{25}+\hat{\beta}_{35}+\hat{\beta}_{45}+\hat{\beta}_{55}+\hat{\beta}_{64}=Y_{11..} = 68$

τ_{12}: $6(\hat{\mu}+\hat{\tau}_{12})+0+\hat{\beta}_{14}+\hat{\beta}_{26}+\hat{\beta}_{36}+\hat{\beta}_{46}+\hat{\beta}_{56}+\hat{\beta}_{64}=Y_{12..} = 57$

τ_{13}: $6(\hat{\mu}+\hat{\tau}_{13})+0+\hat{\beta}_{15}+\hat{\beta}_{24}+\hat{\beta}_{36}+\hat{\beta}_{44}+\hat{\beta}_{56}+\hat{\beta}_{65}=Y_{13..} = 56$

τ_{14}: $6(\hat{\mu}+\hat{\tau}_{14})+0+\hat{\beta}_{15}+\hat{\beta}_{25}+\hat{\beta}_{34}+\hat{\beta}_{45}+\hat{\beta}_{54}+\hat{\beta}_{65}=Y_{14..} = 66$

τ_{15}: $6(\hat{\mu}+\hat{\tau}_{15})+0+\hat{\beta}_{15}+\hat{\beta}_{26}+\hat{\beta}_{35}+\hat{\beta}_{46}+\hat{\beta}_{55}+\hat{\beta}_{65}=Y_{15..} = 65$

τ_{16}: $6(\hat{\mu}+\hat{\tau}_{16})+0+\hat{\beta}_{16}+\hat{\beta}_{24}+\hat{\beta}_{35}+\hat{\beta}_{44}+\hat{\beta}_{55}+\hat{\beta}_{66}=Y_{16..} = 63$

τ_{17}: $6(\hat{\mu}+\hat{\tau}_{17})+0+\hat{\beta}_{16}+\hat{\beta}_{25}+\hat{\beta}_{36}+\hat{\beta}_{45}+\hat{\beta}_{56}+\hat{\beta}_{66}=Y_{17..} = 65$

τ_{18}: $6(\hat{\mu}+\hat{\tau}_{18})+0+\hat{\beta}_{16}+\hat{\beta}_{26}+\hat{\beta}_{34}+\hat{\beta}_{46}+\hat{\beta}_{54}+\hat{\beta}_{66}=Y_{18..} = 62$

Instead of setting up the normal equations for the β_{jg}'s, an interaction table will be constructed and the β_{jg}'s estimated as before

Block No. (j=1 or 6)	Arrangement I (totals) Rep 1	Rep 6	Sum	Difference	Block No. (j=2 or 4)	Arrangement II (totals) Rep 2	Rep 4	Sum	Difference
j1	18	18	36	0	j1	23	35	58	-12
j2	17	26	43	-9	j2	20	29	49	- 9
j3	42	33	75	9	j3	25	49	74	-24
j4	22	31	53	-9	j4	22	34	56	-12
j5	33	24	57	9	j5	33	45	78	-12
j6	30	30	60	0	j6	21	42	63	-21
Sum	162	162	324	0	Sum	144	234	378	-90
Mean	9	9	9	0	Mean	8	13	10.5	-

Block No. (j=3 or 5)	Arrangement III			
	Rep 3	Rep 5	Sum	Difference
j1	32	20	52	12
j2	49	31	80	18
j3	32	17	49	15
j4	43	22	65	21
j5	42	33	75	9
j6	36	21	57	15
Sum	234	144	378	90
Mean	13	8	10.5	-

The $Q_{.jg}$ (j=I, II or III) values computed for each arrangement are:

$$Q_{.I1} = Y_{.11}+Y_{.61}-2k(\bar{y}_{.I.}-\bar{\bar{y}})-k(\bar{y}_{.6.}-\bar{y})-2\sum_{i=1}^{v} n_{iI1}\bar{y}_{i..}$$

$$= 36-3(9-10+9-10)-2/6(50+45+39)=42-134/3 = -8/3$$

$$Q_{.I2} = 43-3(-2)-2/6(32+26+91) = -2/3$$

$$Q_{.I3} = 75-3(-2)-1/3(85+75+73) = 10/3$$

$$Q_{.I4} = 53-3(-2)-1/3(62+68+57) = -10/3$$

$$Q_{.I5} = 57+6-1/3(56+66+65) = 2/3$$

$$Q_{.I6} = 60+6-1/3(63+65+62) = 8/3$$

Similarly,

$Q_{.II1} = 55-167/3 = -2/3$

$Q_{.II2} = 46-146/3 = -8/3$

$Q_{.II3} = 71-203/3 = 10/3$

$Q_{.II4} = 53-181/3 = -22/3$

$Q_{.II5} = 75-199/3 = 26/3$

$Q_{.II6} = 60-184/3 = -4/3$

$Q_{.III1} = 49-149/3 = -2/3$

$Q_{.III2} = 77-221/3 = 10/3$

$Q_{.III3} = 46-146/3 = -8/3$

$Q_{.III4} = 62-190/3 = -4/3$

$Q_{.III5} = 72-196/3 = 20/3$

$Q_{.III6} = 54-178/3 = -16/3$

Likewise, the $\hat{\beta}_{jg}$ (j=I, II, or III) are:

$$\hat{\beta}_{I1} = \frac{1}{qv}\left\{3kQ_{\bullet I1}\right\} + \frac{1}{k(k+1)}\sum_{i=1}^{k^2}(\bar{y}_{i\bullet}-\bar{y})$$

$$= 1/4(-8/3)+1/12(516/6-90) = -1$$

$$\hat{\beta}_{I2} = 1/4(-2/3)-1/3 = -1/2$$

$$\hat{\beta}_{I3} = 1/4(10/3)-1/3 = 1/2$$

$$\hat{\beta}_{I4} = 1/4(-10/3)+1/12(564/6-90) = -1/2$$

$$\hat{\beta}_{I5} = 1/4(2/3)+1/3 = 1/2$$

$$\hat{\beta}_{I6} = 1/4(8/3)+1/3 = 1$$

Similarly,

$\hat{\beta}_{II1} = -1/2$

$\hat{\beta}_{II2} = -1$

$\hat{\beta}_{II3} = 1/2$

$\hat{\beta}_{II4} = -3/2$

$\hat{\beta}_{II5} = 5/2$

$\hat{\beta}_{II6} = 0$

$\hat{\beta}_{III1} = -1/2$

$\hat{\beta}_{III2} = 1/2$

$\hat{\beta}_{III3} = -1$

$\hat{\beta}_{III4} = 0$

$\hat{\beta}_{III5} = 2$

$\hat{\beta}_{III6} = -1$

In the above $\beta_{Ig}=1/2(\beta_{1g}+\beta_{6g})$, $\beta_{IIg}=1/2(\beta_{2g}+\beta_{4g})$, and $\beta_{IIIg}=1/2(\beta_{3g}+\beta_{5g})$ where the β_{jg} are the parameters used to construct the example. With these values of the $\hat{\beta}_{jg}$'s we are now in a position to estimate the τ_i.

$$\hat{\tau}_1 = \bar{y}_{1\circ\bullet}-\bar{y}-1/6(2\beta_{I1}+2\beta_{II1}+2\beta_{III1})$$

$$= (50-60-2(-1-1/2-1/2))/6 = -1$$

$$\hat{\tau}_2 = (45-60-2(-1-1+1/2))/6 = -2$$

Similarly,

		$Q_{1\cdot\cdot} = 4/3$	$Q_{2\cdot\cdot} = -30/3$
$\hat{\tau}_3 = -3$	$\hat{\tau}_{11} = 0$	$Q_{3\cdot\cdot} = -42/3$	$Q_{11\cdot\cdot} = -2/3$
$\hat{\tau}_4 = -4$	$\hat{\tau}_{12} = 0$	$Q_{4\cdot\cdot} = -54/3$	$Q_{12\cdot\cdot} = -2/3$
$\hat{\tau}_5 = -5$	$\hat{\tau}_{13} = 0$	$Q_{5\cdot\cdot} = -66/3$	$Q_{13\cdot\cdot} = -2/3$
$\hat{\tau}_6 = 5$	$\hat{\tau}_{14} = 0$	$Q_{6\cdot\cdot} = 76/3$	$Q_{14\cdot\cdot} = -2/3$
$\hat{\tau}_7 = 4$	$\hat{\tau}_{15} = 0$	$Q_{7\cdot\cdot} = 42/3$	$Q_{15\cdot\cdot} = 0$
$\hat{\tau}_8 = 3$	$\hat{\tau}_{16} = 0$	$Q_{8\cdot\cdot} = 52/3$	$Q_{16\cdot\cdot} = -2/3$
$\hat{\tau}_9 = 2$	$\hat{\tau}_{17} = 0$	$Q_{9\cdot\cdot} = 18/3$	$Q_{17\cdot\cdot} = 0$
$\hat{\tau}_{10} = 1$	$\hat{\tau}_{18} = 0$	$Q_{10\cdot\cdot} = 12/3$	$Q_{18\cdot\cdot} = -2/3$

The sums of squares in the analysis of variance are computed as:

<u>Total with 107 degrees of freedom</u>

$$\sum_{i,j,g} Y^2_{ijg} - \frac{Y^2_{\cdot\cdot\cdot}}{108=2(3)(2k^2)} = 12302 - 10800 = 1502 \quad .$$

<u>Replicate with 5 degrees of freedom</u>

$$\sum_{j=1}^{6} \frac{Y^2_{\cdot j\cdot}}{v} - \frac{Y^2_{\cdot\cdot\cdot}}{rv} = 11304 - 10800 = 504 \quad .$$

<u>Treatments (ignoring blocks) with 17 degrees of freedom</u>

$$\sum_{i=1}^{18} \frac{Y^2_{i\cdot\cdot}}{r} - \frac{Y^2_{\cdot\cdot\cdot}}{rv} = 11613 - 10800 = 813 \quad .$$

<u>Blocks (ignoring treatments) within replicates with 30 d.f.</u>

$$\sum_{j=1}^{r} \left\{ \sum_{g=1}^{2k} \frac{Y^2_{\cdot jg}}{k} - \frac{Y^2_{\cdot j\cdot}}{2k^2} \right\} = 11792\tfrac{2}{3} - 11304 = \frac{1466}{3} \quad .$$

Treatments 1 to 9 vs. 10 to 18 with 1 d.f.

$$\frac{(516-564)^2}{6(9)(1+1)=108} = \frac{48^2}{108} = \frac{64}{3} \quad .$$

Treatments 1 to 9 vs. 10 to 18 x replicates with 5 d.f.

$$\frac{(77-85)^2+(68-76)^2+(113-121)^2+(68-76)^2 + (113-121)^2+(77-85)^2}{18}$$

$$- \frac{(516-564)^2}{108} = 0 \quad .$$

Blocks (eliminating treatment effects) within groups with 24 d.f.

$$\sum_{j=I}^{III} \left\{ \sum_{g=1}^{3} \hat{\beta}_{jg} Q_{\cdot jg} - \sum_{g=1}^{3} \hat{\beta}_{jg} \sum_{g=1}^{3} Q_{\cdot jg}/3 + \sum_{g=4}^{6} \hat{\beta}_{jg} Q_{\cdot jg} - \sum_{g=4}^{6} \hat{\beta}_{jg} \sum_{g=4}^{6} Q_{\cdot jg}/3 \right\}$$

= component (b) sum of squares = 70 with 12 d.f.

Interaction of Blocks with reps within arrangements I, II, and III = component (a) =

$$\frac{0^2+(-9)^2+\cdots+9^2+15^2}{6=qk} - \frac{0^2+(-90)^2+90^2}{2qk^2=36} = \frac{3294}{6} - \frac{16200}{36} = 99 \quad .$$

Therefore, blocks (elim. tr.) sum of squares is

70+99=169 with 24 d.f.

Treatment (eliminating blocks) with 16 d.f.

$$\Sigma\hat{\tau}_i Q_{i\cdot\cdot} = \frac{1480}{3}$$

Intrablock error with 56 d.f.

1502-504-813-169 = 16 .

The above sums of squares are summarized in the following analysis of variance table:

Source of variation	d.f.		Sum of squares		Mean square
Replicates	5		504		100.8
Treatments 1 to 9 vs 10 to 18	1		64/3		64/3
Error (a)	5		0		0
Treatments within groups (ignoring blocks)	16		2375/3		49.5
Blocks (elim. treat.) within groups	24		169		7.0417
component (a)		12		99	
component (b)		12		70	
Intrablock error	56		16		.2857
Correction for mean	1		10800		--
Total (uncorrected)	108		12302		--
Treatments (elim. blocks)	16		1480/3		30.833
Blocks (ignoring treatments)	30		1466/3		--